CALCULATION OF THE TEMPERATURES AND THE HEAT BALANCE DURING CONTACT HEAT TRANSFER IN A SYSTEM OF BODIES

N. V. Diligenskii and Yu. I. Ivanov

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A method is proposed for calculating the mean-integral contact temperatures and the heat balance during complex heat transfer between four bodies in contact.

We consider a system of four bodies in contact: bodies 1, 2, 3 are rotating and the semiinfinite body 4 is stationary (Fig. 1). Bodies 1 and 2 are in contact across area 1, bodies 2 and 3 are in contact across area 2, and bodies 2 and 4 are in contact across area 3.

At the contact areas there is heat dissipation at a uniform rate.

The problem will be treated as two-dimensional for bodies 1, 2, and 3, i.e., we will operate with mean-over-the-width (l) temperatures at the contact areas. The length of all contact areas (b_k) will be assumed the same ($b_1 = b_2 = b_3 = b$). Bodies 1, 2, and 3 are cooled uniformly at their end surfaces. The thermophysical characteristics of the bodies are independent of the temperature.

With all these stipulations the problem can be formulated in terms of the following equations:

$$\mathrm{Pe}_k \frac{\partial \theta_k}{\partial \varphi} - \frac{\partial^2 \theta_k}{\partial \rho^2} - \frac{1}{\rho} \cdot \frac{\partial \theta_k}{\partial \rho} - \frac{1}{\rho^2} \cdot \frac{\partial^2 \theta_k}{\partial \varphi^2} + \mathrm{Bi}_k \theta_k = \frac{\mathrm{Ki}_{km}}{\Lambda_k \beta_k} \varkappa(0, \beta_k)\, \delta(\rho - 1), \quad k = 1,3, \quad m = 1,2, \tag{1}$$

$$\mathrm{Pe}_2 \frac{\partial \theta_2}{\partial \varphi} - \frac{\partial^2 \theta_2}{\partial \rho^2} - \frac{1}{\rho} \cdot \frac{\partial \theta_2}{\partial \rho} - \frac{1}{\rho^2} \cdot \frac{\partial^2 \theta_2}{\partial \varphi^2} + \mathrm{Bi}_2 \theta_2 \tag{2}$$

$$= \frac{\delta(\rho - 1)}{\Lambda_2 \beta_2} \left[\mathrm{Ki}_{21} \varkappa(0;\ \beta_2) + \mathrm{Ki}_{23} \varkappa\left(\frac{\pi}{2};\ \frac{\pi}{2} + \beta_2\right) + \mathrm{Ki}_{22} \varkappa(\pi;\ \pi + \beta_2) \right], \tag{3}$$

$$\frac{\partial^2 \theta_4}{\partial x^2} + \frac{\partial^2 \theta_4}{\partial y^2} + \frac{\partial^2 \theta_4}{\partial z^2} = \frac{\mathrm{Ki}_{43}}{\Lambda_4} \gamma(x, y)\, \delta(z).$$

Here $\delta(x)$ is the Dirac δ-function and

$$\varkappa(a, c) = \begin{cases} 0 & \varphi < a, \\ & \varphi > c, \\ 1 & a < \varphi < c, \end{cases}$$

$$\gamma(x, y) = \begin{cases} 1 & -\frac{1}{2} < x < \frac{1}{2},\ 0 < y < l, \\ 0 & |x| > \frac{1}{2}, \\ & y > l,\ y < 0. \end{cases}$$

The boundary conditions for Eqs. (1)-(3) are:

$$\left.\frac{\partial \theta_k}{\partial \rho}\right|_{\rho=0} = 0, \quad k = 1, 2, 3, \tag{4}$$

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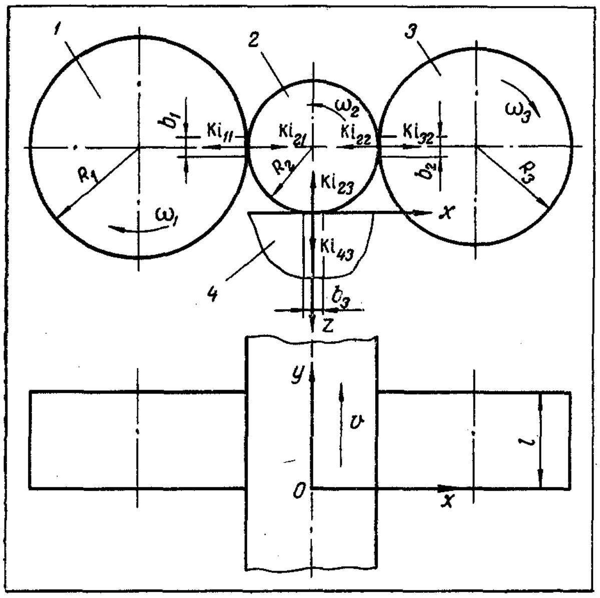

Fig. 1. Schematic representation of bodies in contact.

$$\theta_4 \to 0 \quad \text{as} \quad x,\ y,\ z \to \infty, \tag{5}$$

$$\int_0^{\beta_1} \theta_1 d\varphi|_{\rho=1} = \int_0^{\beta_2} \theta_2 d\varphi|_{\rho=1}, \tag{6}$$

$$\int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\beta_2} \theta_2 d\varphi|_{\rho=1} = \frac{1}{l} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \int_0^{l} \theta_4 dxdy|_{z=0}, \tag{7}$$

$$\int_{\pi}^{\pi+\beta_2} \theta_2 d\varphi|_{\rho=1} = \int_0^{\beta_3} \theta_3 d\varphi|_{\rho=1}, \tag{8}$$

$$\mathrm{Ki}_1 = \mathrm{Ki}_{11} + \mathrm{Ki}_{21},\ \mathrm{Ki}_2 = \mathrm{Ki}_{32} + \mathrm{Ki}_{22},\ \mathrm{Ki}_3 = \mathrm{Ki}_{43} + \mathrm{Ki}_{23}. \tag{9}$$

Equating the temperature of different bodies at their contacts and considering the heat balance at the contact surfaces, we will set up a system of algebraic equations for the thermal fluxes transmitted to the contacting bodies and then solve it.

The solution to Eq. (1) can be represented as a sum of two components: a constant component θ_k equal to the mean-integral solution with respect to angle φ

$$\bar{\theta}_k(\rho) = \frac{1}{2\pi} \int_0^{2\pi} \theta_k(\rho, \varphi)\, d\varphi \tag{10}$$

and an alternating component

$$\tilde{\theta}_k(\rho, \varphi) = \theta_k(\rho, \varphi) - \bar{\theta}_k(\rho). \tag{11}$$

Integration over angle φ from 0 to 2π yields the following equation for the constant component:

$$-\frac{\partial^2 \bar{\theta}_k}{\partial \rho^2} - \frac{1}{\rho} \cdot \frac{\partial \bar{\theta}_k}{\partial \rho} + \mathrm{Bi}_k \bar{\theta}_k = \frac{\mathrm{Ki}_m\, \delta(\rho - 1)}{2\pi\Lambda_k}. \tag{12}$$

The solution to (12) is

$$\bar{\theta}_k = \frac{\mathrm{Ki}_m}{2\pi\Lambda_k} \cdot \frac{I_0(\rho\sqrt{\mathrm{Bi}_k})}{\sqrt{\mathrm{Bi}_k}\, I_1(\sqrt{\mathrm{Bi}_k})}, \tag{13}$$

where I_0 and I_1 denote the modified zeroth- and first-order Bessel functions.

At high rotational speeds, as has been shown in [1], the alternating component penetrates only slightly into the bodies and is little affected by the heat transfer. An asymptotic representation of the alternating component at a high speed is given in [2]. Neglecting all but the first term of the expansion at small contact angles, the alternating component of the surface temperature may be written in the form:

$$\tilde{\theta}_k = \frac{2\mathrm{Ki}_m \sqrt{\varphi}}{\Lambda_k \beta_k \sqrt{\pi \mathrm{Pe}_k}}, \quad \varphi \in [0, \beta_k]. \tag{14}$$

The next term in the asymptotic expansion (14) is of the order O in 1/Pe. Practical experience has shown that, when Pe > 20, calculations based on the first term only are sufficiently accurate.

Expression (14) is the solution to the equation:

$$\mathrm{Pe}_k \frac{\partial \theta_k}{\partial \varphi} - \frac{\partial^2 \theta_k}{\partial \rho^2} = \frac{\mathrm{Ki}_m}{\Lambda_k \beta_k} \delta(1 - \rho) . \tag{15}$$

A calculation of the mean-integral contact temperature yields

$$\theta_{km} = \frac{\mathrm{Ki}_{km} \mathrm{I}_0 (\sqrt{\mathrm{Bi}_k})}{2\pi \Lambda_k \sqrt{\mathrm{Bi}_k} \mathrm{I}_1 (\sqrt{\mathrm{Bi}_k})} + \frac{4\mathrm{Ki}_{km}}{3\Lambda_k \sqrt{\pi \mathrm{Pe}_k \beta_k}}, \tag{16}$$

where k = 1, 3 and m = 1, 2. Let us assume that

$$A_k = \frac{\mathrm{I}_0 (\sqrt{\mathrm{Bi}_k})}{2\pi \Lambda_k \sqrt{\mathrm{Bi}_k}\, \mathrm{I}_1 (\sqrt{\mathrm{Bi}_k})}, \quad B_k = \frac{4}{3\Lambda_k \sqrt{\pi \mathrm{Pe}_k \beta_k}},$$

$$Д_k = A_k + B_k.$$

Analogously, for the second body we have

$$\theta_{2m} = \frac{(\mathrm{Ki}_{21} + \mathrm{Ki}_{22} + \mathrm{Ki}_{23}) \mathrm{I}_0 (\sqrt{\mathrm{Bi}_2})}{2\pi \Lambda_2 \sqrt{\mathrm{Bi}_2}\, \mathrm{I}_1 \sqrt{(\mathrm{Bi}_2)}} + \frac{4\mathrm{Ki}_{2m}}{3\Lambda_2 \sqrt{\pi \mathrm{Pe}_2 \beta_2}}, \tag{17}$$

where m = 1, 2, 3.

We now introduce the following designations:

$$A_2 = \frac{\mathrm{I}_0 (\sqrt{\mathrm{Bi}_2})}{2\pi \Lambda_2 \sqrt{\mathrm{Bi}_2}\, \mathrm{I}_1 (\sqrt{\mathrm{Bi}_2})}, \quad B_2 = \frac{4}{3\Lambda_2 \sqrt{\pi \mathrm{Pe}_2 \beta_2}}, \quad Д_2 = A_2 + B_2.$$

The solution for the temperature of a rectangular area through which a uniform thermal flux passes into the semiinfinite body will be taken from [3]:

$$\theta_4 = \frac{\mathrm{Ki}_{43}}{\Lambda_4 \pi l b^2} \left\{ b l^2 \,\mathrm{Arsh} \frac{b}{l} + l b^2 \,\mathrm{Arsh} \frac{l}{b} + \frac{1}{3} [b^2 + l^2 - (b^2 + l^2)^{3/2}] \right\}. \tag{18}$$

It will also be assumed that

$$Д_4 = \frac{1}{\Lambda_4 \pi l b^2} \left\{ b l^2 \,\mathrm{Arsh} \frac{b}{l} + l b^2 \,\mathrm{Arsh} \frac{l}{b} + \frac{1}{3} [b^2 + l^2 - (b^2 + l^2)^{3/2}] \right\}.$$

Substituting from Eqs. (16)-(18) into Eqs. (6)-(9) yields a system of algebraic equations, which will be written down in the following form:

$$\begin{aligned}
(\mathrm{Ki}_1 - \mathrm{Ki}_{21})\, Д_1 &= (\mathrm{Ki}_{21} + \mathrm{Ki}_{22} + \mathrm{Ki}_{23})\, A_2 + \mathrm{Ki}_{21} \cdot B_2, \\
(\mathrm{Ki}_2 - \mathrm{Ki}_{22})\, Д_3 &= (\mathrm{Ki}_{21} + \mathrm{Ki}_{22} + \mathrm{Ki}_{23})\, A_2 + \mathrm{Ki}_{22} \cdot B_2, \\
(\mathrm{Ki}_3 - \mathrm{Ki}_{23})\, Д_4 &= (\mathrm{Ki}_{21} + \mathrm{Ki}_{22} + \mathrm{Ki}_{23})\, A_2 + \mathrm{Ki}_{23} \cdot B_2.
\end{aligned} \tag{19}$$

From here we determine Ki_{21}, Ki_{22}, and Ki_{23} with the aid of the Cramer rules.

With the dimensionless thermal fluxes Ki_{21}, Ki_{22}, and Ki_{23} known, we determine the contact temperatures with the aid of Eqs. (16)-(18).

The fractions of the heat flowing into contacting bodies 1-4 are determined as follows:

$$\eta_1 = \frac{\mathrm{Ki}_1 - \mathrm{Ki}_{21}}{\Sigma \mathrm{Ki}_m}, \quad \eta_2 = \frac{\mathrm{Ki}_{21} + \mathrm{Ki}_{23} + \mathrm{Ki}_{22}}{\Sigma \mathrm{Ki}_m},$$
$$\eta_3 = \frac{\mathrm{Ki}_2 - \mathrm{Ki}_{22}}{\Sigma \mathrm{Ki}_m}, \quad \eta_4 = \frac{\mathrm{Ki}_3 - \mathrm{Ki}_{23}}{\Sigma \mathrm{Ki}_m}. \tag{20}$$

In solving the problem, we consider the heat transfer between the outer surface of cylindrical bodies and the surrounding atmosphere. The alternating component may here be considered invariable, and the equation for the constant (mean-integral) component will be

$$\frac{\partial^2\theta_k}{\partial\rho^2} + \frac{1}{\rho}\cdot\frac{\partial\theta_k}{\partial\rho} = 0,$$
$$\left.\frac{\partial\theta_k}{\partial\rho}\right|_{\rho=1} = \mathrm{Ki}_m - \mathrm{Bi}_k\theta_k|_{\rho=1}, \tag{21}$$
$$\left.\frac{\partial\theta_k}{\partial\rho}\right|_{\rho=0} = 0.$$

The solution to Eq. (21) is $\theta_k = \mathrm{Ki}_m/\mathrm{Bi}_k$, from which

$$A_k = \frac{1}{2\pi\Lambda_k \mathrm{Bi}_k}.$$

If body 2, besides rotating, is also in translatory motion along the y-axis at a velocity v, this can be accounted for by the appropriate relation between A_2 and velocity v. At a high velocity,

$$A_2 = \frac{2a_2\sqrt{l}}{3\pi R_2\Lambda_2\sqrt{\pi v a_2}}.$$

The solution obtained here may be used for analyzing heat transfer and contact temperatures in numerous technological processes. The basic schematic diagram (Fig. 1) coincides with the mathematical model for an analysis of thermal effects during centerless surface grinding and cutting grinding. With the number of contacting bodies reduced to two, we have a mathematical model of circular grinding.

NOTATION

$\theta_k = (t-t_0)/(t_s-t_0)$	are dimensionless temperatures of the bodies (k = 1-4);
k	is the number of the body;
m	is the number of the contact area;
t_0	is the ambient temperature;
t	is the body temperature;
t_s	is the scale temperature;
$\mathrm{Pe}_k = \omega_k R_k^2/a_k$	is the Peclet number;
ω_k	are the angular velocities of the bodies;
R_k	are the radii of the bodies;
a_k	are the thermal diffusivities of the bodies;
φ	is the angular coordinate;
$\rho = r/R_k$	is a dimensionless radial coordinate;
r	is the radius at any point;
$\mathrm{Bi}_k = 2\alpha_k R_k^2/\lambda_k l$	is the Biot number (k = 1, 2, 3);
α_k	are the coefficients of superficial heat transfer;
λ_k	are the thermal conductivities of the bodies;
l	is the length of bodies 1 and 2;
$\Lambda_k = \lambda_k/(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)$	are the dimensionless thermal conductivities;
x, y, z	are dimensionless coordinates;
$\beta_k = b/R_k$	are the relative lengths of the contact areas;
$\mathrm{Ki}_m = q_m b/\Sigma\lambda_k(t_s-t_0)$	is the dimensionless thermal flux at contact area m (m = 1, 2, 3);
Ki_{km}	is the dimensionless thermal flux passing through m-th contact area into k-th body;
q_m	is the quantity of heat at contact area m (m = 1, 2, 3);
b	is the length of contact sites.

LITERATURE CITED

1. I. C. Lager, "Some problems involving linear sources in heat conduction," Phil. Mag., 35, 169 (1944).
2. N. V. Diligenskii and Yu. P. Kamaev, "On the thermophysics of the grinding process," Zh. Fiz. Khim. Obrabotki Metal., No. 1 (1969).
3. G. Carslaw and D. Yaeger, Thermal Conductance of Solids [Russian translation], Nauka (1964).